1D Cauchy.

$u: \mathbb{R}\times[0,T) \to \mathbb{R}$

$u_{tt} = u_{xx}$ in $\mathbb{R}\times[0,T)$

$u(x,0) = g(x) \in C^2(\mathbb{R})$

$u_t(x,0) = h(x) \in C^1(\mathbb{R})$

d'Alembert formula.

$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

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$$u(x,0) = \frac{1}{2}\left(g(x)+g(x)\right) + \cancel{\frac{1}{2}\int_x^x h}^{\,0} = g(x)$$

$$u_t(x,t) = \frac{1}{2}\left(g'(x+t) - g'(x-t)\right) + \frac{1}{2}\left(h(x+t) + h(x-t)\right)$$

$$u_t(x,0) = \frac{1}{2}\left(\cancel{g(x)} - \cancel{g(x)}\right) + \frac{1}{2}\left(h(x)+h(x)\right) = h(x)$$

$$u_{tt} = \frac{1}{2}\left(g''(x+t) + g''(x-t)\right) + \frac{1}{2}\left(h'(x+t) - h'(x-t)\right)$$

$$u_x = \frac{1}{2}\left(g'(x+t) + g'(x-t)\right) + \frac{1}{2}\left(h(x+t) - h(x-t)\right)$$

$$u_{xx} = \frac{1}{2}\left(g''(x+t) + g''(x-t)\right) + \frac{1}{2}\left(h'(x+t) - h'(x-t)\right)$$

$\therefore u_{tt} = u_{xx}$.

Thm) $g \in C^2$, $h \in C^1$. Then

$$u(x,t) = \frac{1}{2}\left(g(x+t) - g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy$$

is the unique $C^2$ solution.

Remark) Here, we do NOT need any condition at $|x| \approx +\infty$.

$u_t = u_{xx}$ , in $\mathbb{R}\times[0,T)$

we need to assume

$$|u(x,t)| \le e^{ax^2}$$

for the uniqueness of solution.

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proof of uniqueness)

$$0 = u_{tt} - u_{xx} = \partial_t^2 u - \partial_x^2 u$$

$$= (\partial_t - \partial_x)(\partial_t + \partial_x)u.$$

Define $v = (\partial_t + \partial_x)u = u_t + u_x$

Then, $(\partial_t - \partial_x)v = v_t - v_x$

$$= \cancel{u_{tt}} + \cancel{u_{tx}} - \cancel{u_{tx}} - \cancel{u_{xx}}$$

$$= 0$$

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c.f. $0 = u_{tt} + u_{tx} - 2u_{xx} = (\partial_t + 2\partial_x)\overset{w}{\overset{\parallel}{\boxed{(\partial_t - \partial_x)u}}}$

$= (\partial_t^2 + \partial_t\partial_x - 2\partial_x^2)u \Rightarrow 0 = v_t + 2v_x.$

$v_t = v_x$.

Define $\alpha(s) = v(x+s, -s)$

$$\alpha'(s) = v_x(x+s, -s) - v_t(x+s, -s) = 0$$

for each $x \in \mathbb{R}$.

$\Rightarrow$ $v(x,t) = v(x+t, 0)$

We define

$$\psi(x) = v(x,0) \in C^1(\mathbb{R}) \quad (\because u \in C^2)$$

Then. $v(x,t) = \psi(x+t)$

$\Rightarrow$ $u_t(x,t) + u_x(x,t) = \psi(x+t)$

Define $\beta(s) = u(x+s, s)$

$$\beta' = u_x(x+s, s) + u_t(x+s, s) = \psi(x+2s)$$

$$u(x+t, t) - u(x, 0) = \beta(t) - \beta(0)$$
$$= \int_0^t \beta'(s)\,ds = \int_0^t \psi(x+2s)\,ds.$$

$$\Rightarrow u(x+t,t) - g(x) = \int_0^t \psi(x+2s)\,ds.$$

$x+t=z$

$$u(z,t) - g(z-t) = \int_0^t \psi(z-t+2s)\,ds.$$

Reparametrize $s = \frac{1}{2}(y-z+t)$

$$\Rightarrow u(z,t) - g(z-t) = \frac{1}{2}\int_{z-t}^{z+t} \psi(y)\,dy$$

$$\therefore u(x,t) = g(x-t) + \frac{1}{2}\int_{x-t}^{x+t} \psi(y)\,dy.$$

$$\Rightarrow u_t(x,t) = -g'(x-t) + \frac{1}{2}\big(\psi(x+t) + \psi(x-t)\big)$$

$$h(x) = u_t(x,0) = -g'(x) + \psi(x)$$

$$\therefore \psi(x) = h(x) + g'(x)$$

$$\Rightarrow u(x,t) = \frac{1}{2}\big(g(x+t) - g(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

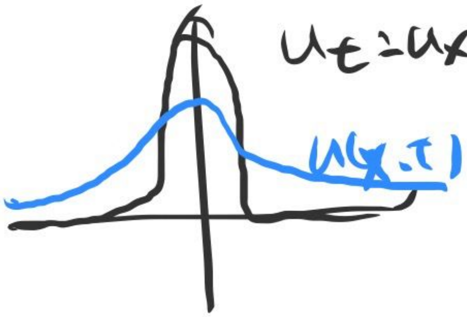
Remark 1) Backward uniqueness

$u: \mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $u_{tt}=u_{xx}$

$\Rightarrow u(x,t)=\frac{1}{2}(g(x+t)-g(x-t))$

$+\frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$

Remark 2) Finite propagation



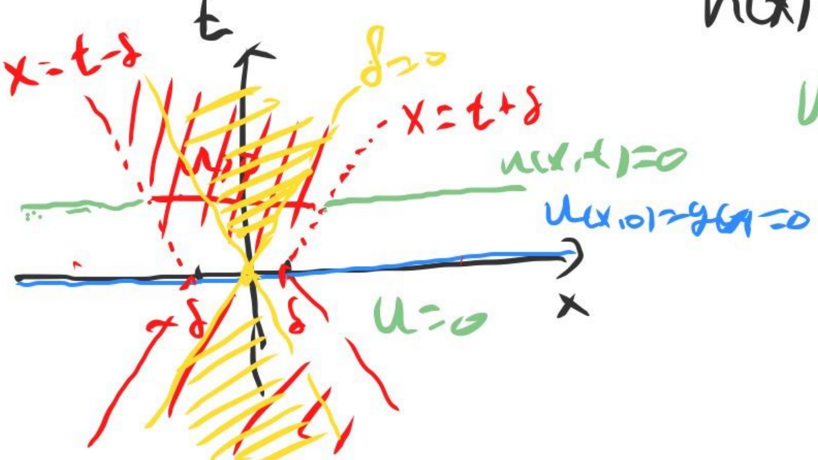

$u_t=u_{xx}$, $u(x,0)=g(x)=0$ if $|x|\geq 1$

$g\geq 0$. $\forall x\in\mathbb{R}$

$\forall t>0$, $u(x,t)>0$. $\forall x\in\mathbb{R}$

Assume. $u_{tt}=u_{xx}$. $g(x)=0$. $h(x)\geq 0$ $\forall x\in\mathbb{R}$

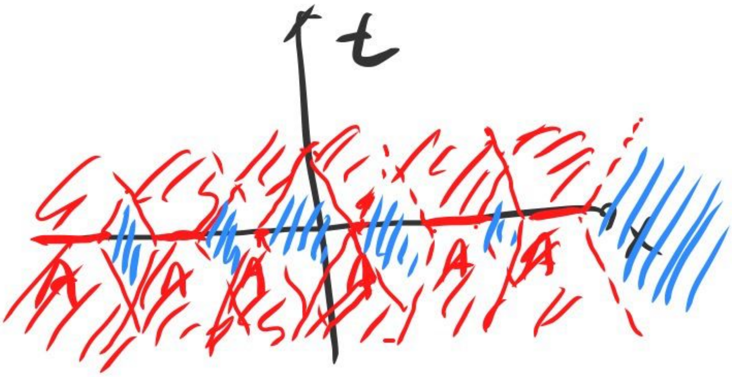
$h(x)=0$ if $|x|\geq\delta>0$.




$u(x,t)=0$

if $x>t+\delta$

or $x<t-\delta$




$g(x), h(x) = 0$

except $A \subset \mathbb{R}$

$u(x,t) = 0$

In the blue region.

by the d'Alembert formula.

Nonhomogeneous equation.

$u: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfies

$$u_{tt} = u_{xx} + f(x,t)$$

$$u(x,0) = g(x), \quad u_t(x,0) = h(x)$$

Then, we can find $u'$ satisfying

$$u'_{tt} = u'_{xx} \quad \text{in } \mathbb{R}\times\mathbb{R}$$

$$u'(x,0) = g(x), \quad u'_t(x,0) = h(x)$$

by using d'Alembert formular.

$\Rightarrow u^2 = u - u'$

$$u^2_{tt} = u^2_{xx} + f(x,t)$$

$$u^2(x,0) = 0, \quad u^2_t(x,0) = 0.$$

Duhamel's method.

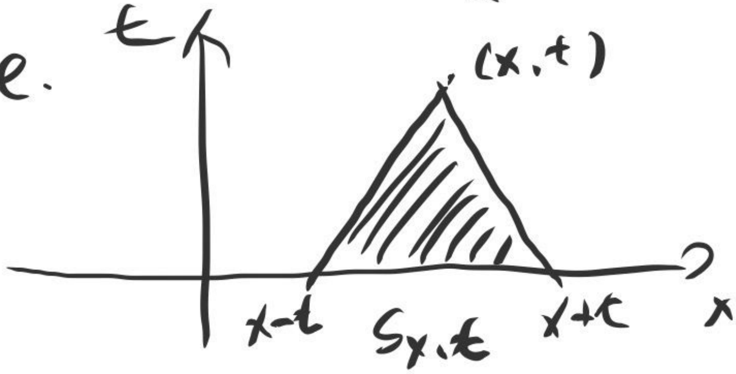
$u_{tt} = u_{xx} + f$. $u(x,0) = u_t(x,0) = 0$

$\exists$ a unique solution

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy\,ds.$$

$$= \frac{1}{2}\int_{S_{x,t}} f(y,s)\,dy\,ds,$$

where.

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we define. $w(x,t;s) = \frac{1}{2}\int_{x-(t-s)}^{x+(t-s)} f(y,s)\,ds$

for $s \in [0,t]$

$\Rightarrow u(x,t) = \int_0^t w(x,t;s)\,ds$ $\left(\begin{array}{l} w(x,t;s) \\ \quad = w^s(x,t) \end{array}\right)$

$$u = \int_0^{t} w(x,t;s)\,ds$$

$$\Rightarrow u_t = w(x,t;t) + \int_0^t w_t(x,t;s)\,ds.$$

$$\left( \because f(t) = \int_0^t g(t,s)\,ds.\right.$$

$$f(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left( \int_t^{t+h} g(t+h,s)\,ds + \int_0^t g(t+h,s) - g(t,s)\,ds \right)$$

$$= \lim \int_t^{t+h} g(t+h,s)\,ds + \int_0^t \lim_{h\to 0} \frac{g(t+h,s) - g(t,s)}{h}\,ds$$

$$\left. = g(t,s) + \int_0^t g_t(t,s)\,ds \right)$$

$$w(x,t;t) = \frac{1}{2}\int_x^x f(y,s)\,dy = 0$$

$$\therefore u_t = \int_0^t w_t(x,t;s)\,ds$$

$$u_{tt} = w_t(x,t;t) + \int_0^t w_{tt}(x,t;s)ds$$

$$w_t(x,t;s) = \frac{1}{2}\begin{pmatrix} f(x+t-s,s) \\ +f(x-t+s,s) \end{pmatrix}$$

$$\Rightarrow w_t(x,t;t) = f(x,t)$$

$$\therefore u_{tt} = f(x,t) + \int_0^t w_{tt}(x,t;s)ds.$$

$$u = \int_0^t w(x,t;s)ds \Rightarrow u_{xx} = \int_0^t w_{xx}(x,t;s)ds.$$

In addstion.

$$w_{xx} = w_{tt} = \frac{1}{2}\left(f_x(x+t-s,s) - f_x(x-t+s,s)\right)$$

$$\therefore u_{tt} = f(x,t) + u_{xx}$$

$$u(x,0) = 0, \quad u_t(x,0) = 0.$$

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Uniqueness: Suppose $u, v$ are such solutions

$\Rightarrow w = u - v$ satisfies $w_{tt} = w_{xx}$. $w(x,0) = w_t(x,0) = 0$

$\Rightarrow w \equiv 0$ (by uniq of wave eq (0)) nD wave, Local energy.

Thm) Let $u \in C^2(\mathbb{R}^n \times [0,T))$ solve

$$u_{tt} = \Delta u.$$

Then, $E(t) = \frac{1}{2} \int_{B_{R-t}(0)} |u_t|^2 + \|\nabla u\|^2 \, dx$

(where $R > 0$ is any constant)

satisfies $E' \le 0$.

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pf) $E'(t) = \frac{-1}{2} \int_{\partial B_{R-t}(0)} |u_t|^2 + \|\nabla u\|^2 \, d\sigma$

$$+ \int_{B_{R-t}(0)} \cancel{u_t u_{tt}} + \nabla u \cdot \nabla u_t \, dx$$

$$\int_{B_{R-t}} \nabla u \cdot \nabla u_t = \int_{\partial B_{R-t}} u_\nu u_t - \cancel{\int_{B_{Rt}} u_t \Delta u}$$

$$\Rightarrow E' = -\frac{1}{2}\int_{\partial B_{R-t}} |u_t|^2 + \|\nabla u\|^2 \, d\sigma$$

$$+ \int_{\partial B_{R-t}} u_\nu u_t \, d\sigma.$$

$$\|\nabla u\|^2 \geq u_\nu^2$$

$$\Rightarrow \frac{1}{2}u_t^2 + \frac{1}{2}\|\nabla u\|^2 \geq \frac{1}{2}u_t^2 + \frac{1}{2}u_\nu^2$$

$$\geq |u_t u_\nu|$$

(by AM-GM)

$$\Rightarrow E' = \int_{\partial B_{R-t}} -\frac{1}{2}u_t^2 - \frac{1}{2}\|\nabla u\|^2 + u_\nu u_t$$

$$\leq 0$$

$\blacksquare$

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Exercise.) If $g, h = 0$ for $|x| \geq M$,

then $u(x,t) = 0$ for $|x| \geq M + |t|$.